THE MASTER EQUATION AND THE EQUATIONS OF QUANTUM DYNAMICS

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We seek in this paper the fundamental form of the equation of quantum dynamics that causes systems to evolve according to the master equation. The direction of this investigation starts out with the master equation taken as an experimental fact and the state of a quantum system described as usual by a vector in Hilbert space. We first prove that the only time evolution operator that will lead to the master equation depends on the exponential of time to the half power. This time dependence suggests that it is due to a random walk. Random walks arise due to a random variable determining the direction taken by each step. On modeling the behavior of the system as a random walk in Hilbert space it is found that the resulting time evolution operator depended on the exponential of time to the half power. This is the same functional form as the one found earlier in the paper that led to the master equation. It is concluded that random time evolution operators describe the underlying process that cause systems to evolve by the master equation.

I. INTRODUCTION

The master equation is used by scientists and engineers to describe and model many different types of non-equilibrium phenomena. Here we take the master equation as an experimental fact. Our object is to find the fundamental physics that leads to this equation.

Two different approaches are taken to derive the master equation. The first, is to start with the dynamic equations of classical or quantum mechanics. Though significant progress has been made in this endeavor, the results of the various attempts are considered by most as unsatisfactory. The second approach is to treat the master equation as the consequence of a stochastic process.

In an earlier paper, [1] we started with a modified form of Dirac's dynamic equation. There we assumed that for a single event the change in the state of the system $|\Psi\rangle$ is given by

$$|\Psi(i+1)\rangle = \exp(-\mathrm{i}A^r)|\Psi(i)\rangle, \tag{1}$$

where A^r is a random Hermitian matrix. In this paper we start with the master equation and work our way in a series of steps to Eq. (1). This path roughly follows the direction taken during the original investigation.

Our intention in this paper is to focus on the underlying physics and at times to provide a more intuitive presentation at the expense of mathematical rigor.

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II. BASIC ASSUMPTIONS

We assume:

1. The master equation [2],

$$\frac{dp_i}{dt} = \sum_j \left[W_{ij} p_j - W_{ji} p_i \right], \tag{2}$$

is an experimental fact.

2. The state of a quantum mechanical system at an instant of time, $\left|\Psi\left(t\right)\right\rangle$, is described by a vector in Hilbert space. That is the description of the state of the system remains unchanged.

The above symbols have their usual meaning.

III: A TIME EVOLUTION OPERATOR THAT LEADS TO THE MASTER EQUATION

The state of the system, $\left|\Psi(t)\right\rangle$, changes according to

$$\left|\Psi(t)\right\rangle = V(t,0)\left|\Psi(0)\right\rangle, \tag{3}$$

where the time evolution operator $V(t,0)$ is a unitary matrix.

Our objective in this section is to find the time evolution operator, $V(t,0)$, that leads to the master equation.

The following derivations largely follow and heavily borrow from our earlier paper [1]. Most importantly we prove here that the sought after time evolution operator can only be a function of the exponential of the time to the half power.

A. Probabilities in terms of expansion coefficients

First we develop the relation between the probabilities, p_i in Eq. (2), the time evolution operator, $V(t,0)$ in Eq. (3), and the state vector $\left|\Psi\right\rangle$'s expansion coefficients C_i.

Consider an isolated system with energy *E*. The state vector $\left|\Psi\right\rangle$ can be expanded in terms of energy eigenkets or basis vectors $\left|a_i\right\rangle$ to give

$$\left|\Psi\right\rangle = \sum_{i=1}^{M} C_i \left|a_i\right\rangle, \tag{4}$$

where *M* is the degree of degeneracy and $C_j = \left\langle a_j \middle| \Psi \right\rangle$ is an expansion coefficient. Eigenkets $\left|a_i\right\rangle$ that have eigenvalues other than *E* are not included in Eq. (4) as the values of their expansion coefficients C_i are zero. The system need not be at equilibrium. In the following we use the eigenkets $\left|a_i\right\rangle$ as a basis that does not change with time. The change in the state of the system is represented by changes in the expansion coefficients.

In the following all summations are taken from one to *M* for all indices unless otherwise indicated.

We are interested with how the probabilities p_i change with time where

$$p_i = C_i^* C_i , \tag{5}$$

and

$$\sum p_i = 1 . \tag{6}$$

The expansion coefficients evolve as a function of the elements of the time evolution operator according to [3] [4]

$$C_i(t) = \sum_j V_{ij} C_j(t_0) , \tag{7}$$

as can also be seen from the following:

$$\left|\Psi(t_0)\right\rangle = \sum_j C_j(t_0) \left|a_j\right\rangle \tag{8}$$

and

$$\left|\Psi(t)\right\rangle = \sum_i C_i(t) \left|a_i\right\rangle . \tag{9}$$

Using Eqs. (3) and (8)

$$\left|\Psi(t)\right\rangle = V(t,t_0) \sum_j C_j(t_0) \left|a_j\right\rangle \tag{10}$$

and

$$\left|\Psi(t)\right\rangle = \sum_i \sum_j V_{ij} C_j(t_0) \left|a_j\right\rangle . \tag{11}$$

On comparing Eqs. (9) and (11), we obtain Eq. (7). See also Tolman [4].

There is a similarity between Eq. (7) and the equation for a change in the basis. One can be thought of as a rotation of a vector in Hilbert space and the other a rotation of the axis making up the basis.

Substituting Eq. (7) into Eq. (5) yields (see also Tolman [4])

$$p_i(t) = p_i^m(t) + p_i^f(t) , \tag{12}$$

where

$$p_i^m(t) = \sum_j V_{ij}^*(t,t_0) V_{ij}(t,t_0)\, p_j(t_0) , \tag{13}$$

and

$$p_i^f(t) = \sum_j \sum_{\substack{k \\ k \neq j}} V_{ik}^*(t,t_0) V_{ij}(t,t_0) C_k^*(t_0) C_j(t_0) . \tag{14}$$

As covered below, the terms p_i^m and p_i^f lead to the master equation and fluctuations, respectively. We consider the p_i^m term in this section through Section V. In Section VI we deal with the fluctuation term p_i^f.

B. Some relations and the principal of the detailed balance

By Stone's Theorem [5] [6] a unitary operator, such as the time evolution operator, can be written in exponential form,

$$V\left(t,t_0\right) = \exp(-iG\Delta s) , \tag{15}$$

where G is a Hermitian operator that is not a function of s and s is a real parameter. (The usual form of this equation has $G = H/\hbar$ [7] [8].) For V to be a function of time, s must be a function of time. We are interested in finding the form of s that will lead to the master equation.

Expanding V and setting

$$F = G^2 \tag{16}$$

we obtain through second order terms (the need to include second order terms is covered in subsection D below)

$$V = 1 - iG\Delta s - \frac{F}{2}\Delta s^2 \tag{17}$$

and

$$V_{ij} = \delta_{ij} - iG_{ij}\Delta s - \frac{F_{ij}}{2}\Delta s^2 . \tag{18}$$

Multiplying V_{ij} by its complex conjugate and retaining only terms through second order (first order terms cancel as $G_{ij} = g_{ij} + ih_{ij}$) gives

$$V_{ij}^* V_{ij} = \delta_{ij} + G_{ij}^* G_{ij}\Delta s^2 - \delta_{ij}\left[\left(F_{ij}^* + F_{ij}\right)/2\right]\Delta s^2 . \tag{19}$$

Defining

$$W_{ij} = G_{ij}^* G_{ij} - \delta_{ij}\left[\left(F_{ij}^* + F_{ij}\right)/2\right], \tag{20}$$

permits us to rewrite Eq. (19) as

$$V_{ij}^* V_{ij} = \delta_{ij} + W_{ij}\Delta s^2 . \tag{21}$$

Using Eq. (20) and because G is Hermitian, $G_{ij}^* = G_{ji}$, we obtain

$$W_{ij} = W_{ji}, \; i \neq j \tag{22}$$

and

$$W_{ij} \geq 0 . \tag{23}$$

Equation (22) is the definition of the principal of the detailed balance.

Because V is unitary

$$\sum_j V_{kj}^* V_{ij} = \delta_{ki} . \tag{24}$$

Substituting Eq. (21) into Eq. (24) and rearranging yields for $k = i$

$$W_{ii} = -\sum_{j, j \neq i} W_{ij} . \tag{25}$$

C. The Master Equation

Substituting Eq. (21) into Eq. (13) we obtain

$$\left(p_i^m(t) - p_i^m(t_0)\right) = \sum_j W_{ij} p_j \left(\Delta s\right)^2 . \tag{26}$$

As $s\left(t_0\right) = 0$

$$\lim_{t \to 0} \frac{\left(\Delta s\right)^2}{\Delta t} = 2s\frac{ds}{dt}, \tag{27}$$

and

$$\frac{dp_i^m}{dt} = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(p_i^m(t) - p_i^m(t_0) \right) = \sum_j W_{ij} p_j 2s \frac{ds}{dt}. \tag{28}$$

Rearranging we obtain

$$\frac{dp_i^m}{dt} = \sum_{j, j \neq i} W_{ij} p_j 2s \frac{ds}{dt} + W_{ii} p_i 2s \frac{ds}{dt}, \tag{29}$$

or using Eq. (25) and as $\left(W_{ii} p_i - W_{ii} p_i \right) = 0$,

$$\frac{dp_i^m}{dt} = \sum_j \left(W_{ij} p_j - W_{ji} p_i \right) 2s \frac{ds}{dt}. \tag{30}$$

For us to obtain the master equation, Eq. (2),

$$2s \frac{ds}{dt} = 1 \tag{31}$$

must hold. Usually, $G = H/\hbar$ and $s = t$. Substituting $s = t$ into Eq. (31) shows that the usual time evolution operator, $V^{usual} = \exp\left(-i H/\hbar \, t \right)$, cannot lead to the master equation.

Integrating Eq. (31) we obtain the unique solution

$$s^2 = t \tag{32}$$

or

$$s = t^{1/2}. \tag{33}$$

Substituting $s = t^{1/2}$ into Eq. (30) yields the master equation, Eq. (2).

We conclude that one must use the time evolution operator

$$V(t, t_0) = \exp\left(-iG(t - t_0)^{1/2} \right), \tag{34}$$

or equivalently

$$V = \exp(-iGt^{1/2}), \tag{35}$$

to obtain the master equation.

D. Second order terms

We included the second order term $\left(F/2 \right) \Delta s^2$ in Eq. (17) in order to avoid a contradiction. Without the second order term in Eq. (20) $W_{ii} = G_{ii}^* G_{ii}$ would be a positive number. On the other hand, W_{ii} would be negative by Eq. (25) as all of the W_{ij}'s on the right hand side are greater than or equal to zero. This problem is eliminated by including the second order term.

IV. RANDOM WALK HYPOTHESIS AND THE TIME EVOLUTION OPERATOR

This section is devoted to the underlying quantum mechanical process that yields the time evolution operator in Eq. (35). Here we use a different approach than presented in [1].

A. The hypothesis

The unitary operator V given in Eq. (35) is a function of $t^{1/2}$. In a wide variety of systems the change in the state or distance covered is a linear function of time. A notable exception is the random walk. There the most likely distance covered (mean square displacement or expectation value) is proportional to the number of steps taken to the half power. This holds for a wide variety of different types of random walks, including walks on high dimensional spaces, walks where the size of the step is given by a random variable and walks that are not on a lattice [9]. In all cases the direction taken by a step is given by a random variable. If the number of steps taken is on average proportional to the time, this suggests that the underlying process leading to Eq. (35) is a random walk.

The above conclusion is made more compelling by the fact that the master equation can be derived by treating it as a stochastic process [10].

The following provides yet another indication that it is worth considering a random walk.

Consider an isolated two dimensional degenerate system. We look at a change in the state of the system two ways. First, using the master equation, Eq. (2), we have

$$\frac{dp_1}{dt} = W_{12} p_2 - W_{21} p_1 \tag{36}$$

and

$$\frac{dp_2}{dt} = W_{21} p_1 - W_{12} p_2 . \tag{37}$$

The probabilities in any transition are subject to the constraint

$$p_1 + p_2 = 1 . \tag{38}$$

A consequence of Eq. (38) is that the first term on the right hand side in Eq. (36) and the second term on the right hand side in Eq. (37) are two sides of the same transition. We can call this the forward transition. The reverse transition consists of the second term in Eq. (36) and the first term in Eq. (37). Both transitions are independent and because $W_{12} = W_{21}$ they have equal probability of occurring.

Second, we hypothesize the following as an alternate way of looking at this same two dimensional system.

The state vector for this system, Eq. (4), is

$$\left|\Psi_{12}\right\rangle = C_1 \left|a_1\right\rangle + C_2 \left|a_2\right\rangle , \tag{39}$$

and

$$p_i = \left|C_i\right|^2 . \tag{40}$$

The change in the state is given by the following equation:

$$\left|\Psi(n+1)\right\rangle = K \left|\Psi(n)\right\rangle , \tag{41}$$

where the time evolution operator K is a random unitary operator equal to the unitary operator U for the forward transition and its transpose $U^{\dagger}$ for the reverse transition. The forward and reverse transitions are independent and have equal probabilities of occurring. The operator U and its transpose are not a function of the state of the system or the index n. The expansion coefficients satisfy before and after a transition

$$\left|C_1\right|^2 + \left|C_2\right|^2 = 1 , \tag{42}$$

and using Eq. (40) we obtain Eq. (38). There are two quantum based equations giving a change in state, one using U for the forward transition, and the other using its transpose for the reverse transition.

The similarity of the master equation and random time evolution operator descriptions is striking. This gives another reason to pursue the random walk hypothesis.

Eq. (41) can be generalized to M dimensions. Before taking this up we further explore the two dimensional case.

B. The two dimensional case

The time evolution operator U describes a rotation of the state vector in Hilbert space. The general form of the rotation operator is complex. See [11] or [12] for a more general form, which if used here would only result in the inclusion of additional constants. For present purposes we use what may be called a base rotation α,

$$U = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}. \tag{43}$$

Substituting $-\alpha$ for α in the above equation yields the transpose of U.

The amount of rotation is determined by the base rotation times the appropriate expansion coefficient. The forward rotation is given by

$$\begin{pmatrix} C_1(n+1) \\ C_2(n+1) \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} C_1(n) \\ C_2(n) \end{pmatrix}. \tag{44}$$

This is an unusual though not unknown type of random walk. It is usual to think of a random walk in terms of a point, or an object that can be represented by a point, moving randomly in some sort of space. There the extent of the movement (which may also be determined by a random variable) is not a function of the position of the point. Here the direction that the state vector moves is governed by the time evolution operator. However, the extent if not direction of the movement, is determined by the state of the system.

The state of the system after N transitions is

$$\left|\Psi(N)\right\rangle = K^N \left|\Psi(0)\right\rangle. \tag{45}$$

We examine the meaning of the terms $\left|\Psi\left(N\right)\right\rangle$ and K^N.

Because we are dealing with a random walk, Eq. (45) describes many possible final states and many possible paths to each of those final states. Each path is made up of the sum of N random steps. Thus the state vector, $\left|\Psi\left(N\right)\right\rangle$, is a random variable that is the sum of N independent random variables. We can apply the central limit theorem [14]. Doing so, we conclude that the scaled distribution of the state vector approaches a normal distribution centered about the expectation value $E\left(\left|\Psi(N)\right\rangle\right)$. As we are dealing with a normal distribution, which is symmetric, all possible state vectors can be grouped in pairs, one on each side of the expectation value. According to the principle of superposition, wherein when a system can be in many states, the state vector $\left|\Psi\left(N\right)\right\rangle$ is the sum of the vectors describing each of those states [17]. Thus each pair of vectors adds to give the state vector. (All vectors in Hilbert space that have the same direction are equivalent irrespective of length.) Combining all vector pairs with state vector, in

the limit of $n \to \infty$, the state vector will have a value equal to the expectation vector. We write $|\Psi(N)\rangle$ in place of $E\left(|\Psi(N)\rangle\right)$.

The term K^N has the following properties: It is a unitary matrix. Though the term is multiplicative, it represents the sum of N α rotations. The rotations making up K^N are of equal magnitude, α, and each has equal probability of being in the forward or reverse direction. This is the description of a symmetric one dimensional random walk [13] [14]. Taking

$$\beta^2 = \left(\sum_{i=1}^{N} \alpha\right)^2, \tag{46}$$

the root mean square displacement or expectation value of ϕ is given by

$$\phi = \sqrt{\overline{\beta^2}} = \alpha\sqrt{N}\,. \tag{47}$$

The expectation value of K^N, is

$$E\left(K^N\right) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}. \tag{48}$$

K^N can take on many different values in addition to its expectation value $E\left(K^N\right)$. That means that more generally it can take on more than one terminal value. But $|\Psi(N)\rangle$ and $|\Psi(0)\rangle$ are non random vectors with definite values. That means that $K(N)$ must also have a definite value. To be consistent with $|\Psi(N)\rangle$ being equal to its expectation value, we take $K(N)$ equal to its expectation value $E\left(K^N\right)$. Eq. (45) has the following meaning:

$$E\left(|\Psi(N)\rangle\right) = E\left(K^N\right)|\Psi(0)\rangle\,. \tag{49}$$

We take the number of transitions N as proportional to the time, t. This is OK except in certain pathological cases. Thus the expectation value of the unitary matrix K is a function of $t^{1/2}$. Eqs. (47) and (48) become, where b is an appropriately scaled constant,

$$\phi = bt^{1/2}, \tag{50}$$

and

$$K(t) = \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix} t^{1/2}. \tag{51}$$

Similarly,

$$|\Psi(t)\rangle = |\Psi(N)\rangle. \tag{52}$$

Eq. (45) becomes

$$|\Psi(t)\rangle = K(t)|\Psi(0)\rangle \tag{53}$$

As $K(t)$ is a unitary matrix, by Stone's theorem

$$K(t) = \exp\left(-iFt^{1/2}\right), \tag{54}$$

where F is a Hermitian matrix. This has the same functional form as Eq. (35), the unitary matrix that leads to the master equation.

C. High dimensional case

One more concept is needed to extend the above to higher dimensional systems. It is introduced here by considering a three dimensional system.

In three dimensions

$$|\Psi\rangle = \sum_{i=1}^{3} C_i |a_i\rangle . \tag{55}$$

Assume that the transitions involving the pairs of expansion coefficients C_1 and C_2, C_2 and C_3, and C_3 and C_1 are independent of each other (this is consistent with the master equation). We call the transitions associated with the C_i and C_j pair of expansion coefficients $i-j$ type transitions.

For each set of $i-j$ type transitions the forward time evolution operator is

$$U_{ij} = \begin{pmatrix} \cos\alpha_{ij} & -\sin\alpha_{ij} \\ \sin\alpha_{ij} & \cos\alpha_{ij} \end{pmatrix}. \tag{56}$$

For clarity we note that for a single transition or step the expansion coefficients evolve according to

$$\begin{pmatrix} C_1(n_{12}+1) \\ C_2(n_{12}+1) \end{pmatrix} = \begin{pmatrix} \cos\alpha_{12} & -\sin\alpha_{12} \\ \sin\alpha_{12} & \cos\alpha_{12} \end{pmatrix} \begin{pmatrix} C_1(n_{12}) \\ C_2(n_{12}) \end{pmatrix}$$

$$\begin{pmatrix} C_2(n_{23}+1 \\ C_3(n_{23}+1) \end{pmatrix} = \begin{pmatrix} \cos\alpha_{23} & -\sin\alpha_{23} \\ \sin\alpha_{23} & \cos\alpha_{23} \end{pmatrix} \begin{pmatrix} C_2(n_{23}) \\ C_3(n_{23}) \end{pmatrix}, \tag{57}$$

$$\begin{pmatrix} C_3(n_{31}+1 \\ C_1(n_{31}+1) \end{pmatrix} = \begin{pmatrix} \cos\alpha_{31} & -\sin\alpha_{31} \\ \sin\alpha_{31} & \cos\alpha_{31} \end{pmatrix} \begin{pmatrix} C_3(n_{31}) \\ C_1(n_{31}) \end{pmatrix}$$

where a_{ij} can equally likely be positive or negative. We note that, say, C_2 can evolve due to $1-2$ type transitions and $2-3$ type transitions. This is similar to how the probabilities evolve in the master equation.

The total number of steps taken N is given by

$$N = N_{12} + N_{23} + N_{31} . \tag{58}$$

Combining the matrices in Eq. (57) we obtain for the random time evolution operator

$$K = \begin{pmatrix} \cos a_{12} + \cos a_{31} & -\sin a_{12} & \sin a_{31} \\ \sin a_{12} & \cos a_{12} + \cos a_{23} & -\sin a_{23} \\ -\sin a_{31} & \sin a_{23} & \cos a_{23} + \cos a_{31} \end{pmatrix}. \tag{59}$$

For $i-j$ type transitions after a random walk of N_{ij} steps we have by the same reasoning as in the two dimensional case

$$\beta_{ij}^2 = \left(\sum_1^{N_{ij}} \alpha_{ij} \right)^2 \tag{60}$$

and

$$\phi_{ij} = \sqrt{\overline{\beta_{ij}^{\,2}}} = \alpha_{ij} \sqrt{N_{ij}} \ . \tag{61}$$

Combining the above gives

$$E\left(K^N\right) = \begin{pmatrix} \cos\phi_{12} + \cos\phi_{31} & -\sin\phi_{12} & \sin\phi_{31} \\ \sin\phi_{12} & \cos\phi_{12} + \cos\phi_{23} & -\sin\phi_{23} \\ -\sin\phi_{31} & \sin\phi_{23} & \cos\phi_{23} + \cos\phi_{31} \end{pmatrix}, \tag{62}$$

where K^N is as before a random time evolution operator.

By the same reasoning as in the previous subsection we have

$$\left|\Psi\left(N\right)\right\rangle = E\left(\left|\Psi\left(N\right)\right\rangle\right), \tag{63}$$

where we now have made use of the multidimensional central limit theorem [15] and the symmetry of the multivariate normal distribution [16]. Further

$$K^N = E\left(K^N\right). \tag{64}$$

Setting N_{ij} proportional to t, Eq. (61) becomes

$$\phi_{ij} = b_{ij} t^{1/2}, \tag{65}$$

and Eq. (62) becomes

$$E\left(K\left(t\right)\right) = \begin{pmatrix} \cos b_{12} + \cos b_{31} & -\sin b_{12} & \sin b_{31} \\ \sin b_{12} & \cos b_{12} + \cos b_{23} & -\sin b_{23} \\ -\sin b_{31} & \sin b_{23} & \cos b_{23} + \cos b_{31} \end{pmatrix} t^{1/2} . \tag{66}$$

Following the reasoning in the last subsection, we obtain Eqs. (53) and (54). As noted there, Eq. (54) has the same functional form as the time evolution operator, Eq. (35), that must be used to obtain the master equation.

The above can readily be extended, at least conceptually, to finding the time evolution operator in higher dimensions. Because more generally $i-j$ type transitions are not correlated with $k-l$ type transitions the logic and details of how this is done remains the same as in the three dimensional case. On doing so we will again obtain Eqs. (53) and (54), the latter of which has the same functional form as Eq. (35), the time evolution operator used to obtain the master equation.

We conclude that the process causing systems to evolve according to the master equation is random quantum transitions that can be described by a random time evolution operator.

V. THE RANDOM TIME EVOLUTION OPERATOR HYPOTHESIS

Above we concluded that the underlying process causing systems to evolve according to the master equation was random quantum transitions. If that conclusion is correct, there should be other examples of this phenomenon. In this section we take a brief look at another example whose behavior can be interpreted in terms of random quantum transitions.

Covering this topic in depth is beyond the scope of this paper. Our objective is to suggest that at least on the surface we are dealing with a similar phenomenon.

Typical of the behavior shown by many systems can be illustrated using the Young two slit experiment [18]. Here a particle, say, an electron or photon, is emitted from a source. The particle passes in some undefined way through two slits and arrives at a sensing screen. If one sends many particles to the screen one finds that they form a pattern of peaks and valleys.

On sending one particle one does not know in advance where it will land. At best one can assign a probability to each position where it might land based on the pattern found when one sent many particles toward the screen. We can think of how to describe where the particle might land, without going into formalities as to how this might be done, as follows: There exists a time evolution operator, or possibly a sequence of time evolution operators, one or more of which are random, which determines where a particle emitted from the source will hit the screen.

Put differently: Where the particle hits the screen is a random variable or event. The location of contact can take on a range of values that are probabilistically determined. Our ultimate goal is to describe this random behavior in terms of an equation that takes the system from one time, say when the particle leaves the emitter, to another, say when the particle hits the screen. A deterministic equation will not fill the bill.

Similar considerations apply to a wide range of systems where scattering and diffraction take place.

As a change in the state of a system can be described by a time evolution operator and as the outcome of a change in the state of many systems are random, it is reasonable to posit that there are random time evolution operators.

VI. FLUCTUATIONS

In Part 1 we found that the probability of the system at time t, $p_i(t)$ given by Eq. (12), was a function of two probabilities. The first, given by Eq. (13), led to the master equation. We now consider the second term given by Eq. (14).

Substituting $s = t^{1/2}$ into Eq. (18), the result into Eq. (14), dropping higher order terms and differentiating (see [1]for details) yields

$$\frac{d^{1/2}}{dt^{1/2}} p_i^f(t) = i \sum_{\substack{l \\ l \neq i}} B_{il} \tag{67}$$

where

$$B_{ij} = G_{il}^* C_l^* C_i - G_{il} C_i^* C_l \,. \tag{68}$$

Fractional differential forms, of which Eq. (67) is one, are associated with chaotic behavior. In [1] we have shown that this equation describes a process that takes a system at equilibrium away from equilibrium. This in turn is counteracted by a process described by the master equation which acts to bring the system back toward equilibrium. The two effects operate on different time scales, $t^{1/2}$ and t. The expected result is chaotic fluctuations around the equilibrium state.

Of particular interest here is that one dynamic equation, Eq. (12), leads to a single equation, Eq. (35) that describes both the master equation and fluctuations. It can be argued that philosophically this is more satisfying than having each phenomenon treated separately.

The description of fluctuations that result from the present theory is an area worth exploring in depth.

IX. SUMMARY

Our goal was to find the underlying process and dynamic equation of quantum mechanics that leads to a system evolving according to the master equation. To find that process the investigation proceeded in a series of steps. The master equation was taken as an experimental fact and the state of the system was described as usual as a vector in Hilbert space. It was then proved that the unitary time evolution leading to the master equation depended on the exponential of time to the half power.

Random walks are systems where the expectation value of the displacement depends on time to the half power. Using this as inspiration, the next step involved modeling the change in the state of the system as a random walk in Hilbert space. Doing so, it was found that the resulting time evolution operator depended on the exponential of time to the half power. This is the same functional form as the one found earlier in the paper that led to the master equation. It was concluded that random time evolution operators describe the underlying process that cause systems to evolve by the master equation.

The two slit Young experiment was briefly examined. One may describe where a single particle hits the screen as the result of a random process. A deterministic equation for how the particle goes from the emitter to the screen will not do.

We concluded that random time evolution operators must be added to the catalogue of quantum mechanical operators. This operator describes the underlying processes that cause a system to evolve by the master equation.

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